

On an integral formula on hypersurfaces in General Relativity

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Abstract

We derive a general integral formula on an embedded hypersurface for general relativistic space-times. Suppose the hypersurface is foliated by two-dimensional compact “sections” S_s . Then the formula relates the rate of change of the divergence of outgoing light rays integrated over S_s under change of section to geometric (convexity and curvature) properties of S_s and the energy-momentum content of the space-time. We derive this formula using the Sparling-Nester-Witten identity for spinor fields on the hypersurface by appropriate choice of the spinor fields. We discuss several special cases which have been discussed in the literature before, most notably the Bondi mass loss formula.

1 Introduction

The purpose of this article is to derive and to discuss an integral formula which can be obtained from the part of the Einstein equations which is “intrinsic” to a hypersurface. This formula contains several special cases which have been discussed before in various different contexts. These are an integral form of the Raychaudhuri equation for null congruences, a special form of the constraints in spherical symmetry and most notably the Bondi mass loss formula. This formula is obtained in the context of two-component spinors and the formalism developed by Szabados [1] on the geometry of embedded two-surfaces. The main ingredient is the Sparling-Nester-Witten identity.

The paper is organised as follows: In section 2 we discuss the necessary background, in particular, the properties of the two-dimensional spin connections known as the Sen connection and the intrinsic spin connection. In section 3 we derive the integral formula and in section 4 we discuss three special cases: the integrated Raychaudhuri equation, spherical symmetry and null infinity. Throughout this work we use the conventions of Penrose and Rindler [2].

2 Geometric preliminaries

Let (M, g) be a Lorentz manifold with spin structure and let Σ be an embedded hypersurface in M . We assume that Σ is diffeomorphic to $S \times I$ where I is an open interval and S is a compact two-dimensional manifold so that $\partial\Sigma = S \times \partial I$. Let s be a parameter in I and assume that the maps $\Psi_s : S \rightarrow \Sigma$ provide the identification of S with its images S_s embedded into Σ . We assume that all these images are spacelike. Then each of these surfaces is the intersection of two unique null hypersurfaces $\mathcal{N}_s^+, \mathcal{N}_s^-$, which exist in a neighbourhood of S_s . At each point of S_s there exist two unique null directions which are orthogonal to the surface. The null hypersurfaces are generated by the null geodesics which start at S_s in these unique null directions. Note, that there is no preferred scaling available on the generators. These null geodesic congruences are (obviously) surface forming.

We can use the null hypersurfaces to introduce coordinates in a neighbourhood of Σ . Choose on Σ a vector field V with $\langle ds, V \rangle = 1$ and coordinates (x^3, x^4) on Σ in such a way that $\langle dx^3, V \rangle = \langle dx^4, V \rangle = 0$, i.e., so that $V = \partial_s$. Now we label each of the null hypersurfaces \mathcal{N}_s^\pm by the value of s on its intersection with Σ . This defines two functions u and v in a neighbourhood of Σ . The two families of null hypersurfaces are obtained by $u = \text{const}$ and $v = \text{const}$, respectively, while Σ is defined by the equation $u = v$. Given a point P near Σ we can now assign coordinates (u, v, x^3, x^4) to it in the following way: P lies on exactly one null hypersurface from each family. These define the values of u and v . (x^3, x^4) are the coordinates of the intersection point with Σ of the generator through P of the null hypersurface $u = \text{const}$. The whole purpose of this discussion was to show that we may assume that the conormals of the null hypersurfaces are gradients. This will be of importance later.

At each point P of S_s the tangent space is the direct sum of the subspace which is tangent to S_s at P and its orthogonal complement. Accordingly, the four-dimensional volume-element e_{abcd} can be split into two parts

$$e_{abcd} = 6 p_{[ab} m_{cd]}, \quad (2.1)$$

where p_{ab} and m_{ab} are two-forms. m_{ab} is the induced area-element on S_s so it annihilates vectors orthogonal to S_s and satisfies $m_{ab} m^{ab} = 2$. Similarly, p_{ab} vanishes when contracted with vectors tangent to the surface and satisfies $p_{ab} p^{ab} = -2$. Moreover, we have $m_{ab} p^{ac} = 0$ and the fact that $m_{ab} = -\frac{1}{2} e_{ab}{}^{cd} p_{cd}$ and $p_{ab} = \frac{1}{2} e_{ab}{}^{cd} m_{cd}$, so m_{ab} and p_{ab} are dual to each other.

In terms of spinors we have $p_{ab} = \frac{1}{2} (\gamma_{AB} \epsilon_{A'B'} + \gamma_{A'B'} \epsilon_{AB})$ with a symmetric spinor γ_{AB} which satisfies $\gamma_{AB} \gamma^{AC} = -\epsilon_B^C$. From the duality we obtain $m_{ab} = i/2 (\gamma_{AB} \epsilon_{A'B'} - \gamma_{A'B'} \epsilon_{AB})$. The “surface spinor” γ_{AB} characterises the properties of the surface in its four-dimensional surroundings in a way similar to the normal vector of a hypersurface. In fact, the flag-pole of any eigenspinor of γ_{AB} (with eigenvalue ± 1) points into one of the null directions defined by S_s .

We want to define two spin connections on the two-surface S_s . First, there is the spin connection induced from the four-dimensional spin connection, the so called two-dimensional Sen connection $\bar{\nabla}$ [1]. Its action on spinor fields is obtained from the four-dimensional spin connection simply by restriction to the

surface:

$$X^a \bar{\nabla}_a \lambda_B = X^a \nabla_a \lambda_B \quad \text{for all } \lambda_B \text{ and all } X^a \text{ tangent to } S_s. \quad (2.2)$$

It contains extrinsic information like the extrinsic curvatures and the torsion of the normal bundle to S_s in M . The other spin connection is as intrinsically defined as possible by the requirements that it agree with the Levi-Civita connection on S_s when acting on vectors which are tangent to S_s and that it annihilate the surface spinor γ_{AB} . These conditions do not completely determine a unique spin connection. Two connections which satisfy the above requirements differ by a term proportional to γ_{AB} , i.e.,

$$\partial_a \lambda_B - \partial'_a \lambda_B = C_a \gamma_{BC} \lambda^C. \quad (2.3)$$

The one-form C_a cannot be fixed by intrinsic properties. However, we can define a spin connection which satisfies these requirements from the two-dimensional Sen connection. First we define the spinor valued one-form $Q_{aB}{}^C = \frac{1}{2} \gamma_A{}^C \bar{\nabla}_a \gamma_B{}^A$ and then the connection ∂ by

$$\partial_a \lambda_B = \bar{\nabla}_a \lambda_B + Q_{aB}{}^C \lambda_C. \quad (2.4)$$

The spin connection thus defined annihilates γ_{AB} and agrees with the intrinsic Levi-Civita connection when acting on tangent vectors of S_s . However, it still contains some extrinsic piece which shows up in its curvature

$$\partial_a \partial_b \lambda_C - \partial_b \partial_a \lambda_C = -S_{abC}{}^E \lambda_E = (\partial^a C_a + \frac{i}{4} \mathcal{R}) m_{ab} \gamma_C{}^E \lambda_E. \quad (2.5)$$

Here the function \mathcal{R} is the scalar curvature of S_s and C_a is a local representation of a $SO(1,1)$ connection in the normal bundle of S_s . This ambiguity does not affect the ensuing discussion, so we will not try to fix C_a any further.

For later use we introduce a spin-frame (o^A, ι^A) adapted to the surface S_s by writing $\gamma_A{}^B = o_A \iota^B + \iota_A o^B$. Thus, o^A and ι^A are eigenspinors of $\gamma_A{}^B$ with eigenvalues -1 and $+1$, respectively. Let l^a, n^a, m^a and \bar{m}^a be the corresponding null tetrad. Then m^a and \bar{m}^a are tangent to S_s while l^a and n^a point along the generators of the null hypersurfaces \mathcal{N}_s^\pm . The spin-frame is defined away from the surface by taking the spinors to be parallel along their respective flag-poles, i.e., we impose the propagation equations $Do^A = 0$ and $D'\iota^A = 0$. This definition yields a spin-frame in a neighbourhood of Σ which is fixed up to the scalings

$$o^A \mapsto c o^A, \quad \iota^A \mapsto \frac{1}{c} \iota^A, \quad (2.6)$$

where c is any function on Σ . This is the natural setup for the GHP-formalism [2, 3] which will be used to some extent in the following discussion. With $e_{abcd} = -24i l_{[a} n_b m_c \bar{m}_{d]}$ the induced area-elements are $m_{ab} = -2im_{[a} \bar{m}_{b]}$ and $p_{ab} = 2l_{[a} n_{b]}$.

In this spin-frame the spinor-valued one-form $Q_{aB}{}^C$ can be determined from the four-dimensional spin connection, with ρ and σ being the usual spin-coefficients

$$Q_{aB}{}^C = -(m_a \rho + \bar{m}_a \sigma) \iota_B \iota^C + (m_a \sigma' + \bar{m}_a \rho') o_B o^C. \quad (2.7)$$

The action of the intrinsic connection on the spin-frame can be found from the fact that it annihilates the surface spinor $\gamma_A{}^B$. For any eigenspinor λ_B of $\gamma_A{}^B$ we have

$$\gamma_A{}^B \lambda_B = \pm \lambda_A \Rightarrow \gamma_A{}^B \partial \lambda_B = \pm \partial \lambda_A \Rightarrow \partial \lambda_A = \Gamma \lambda_A \quad (2.8)$$

for some one-form Γ . Therefore, we obtain

$$\begin{aligned} \partial o_A &= \Gamma o_A, \\ \partial \iota_A &= -\Gamma \iota_A. \end{aligned} \quad (2.9)$$

The one-form Γ can be expanded in terms of the complex basis (m_a, \bar{m}_a) with the coefficients given by the usual spin-coefficients: $\Gamma_a = -m_a \alpha - \bar{m}_a \beta$. This one-form represents both the extrinsic $SO(1,1)$ connection and the intrinsic $SO(2)$ connection which are combined in Γ as (twice) its real and imaginary parts respectively.

For any spinor field $\lambda_A = \lambda_1 o_A - \lambda_0 \iota_A$ we have the formula

$$\partial_a \lambda_A = -(m_a \delta' \lambda_1 + \bar{m}_a \bar{\delta} \lambda_1) o_A + (m_a \delta' \lambda_0 + \bar{m}_a \bar{\delta} \lambda_0) \iota_A, \quad (2.10)$$

and similar expressions for spinor fields with other valences.

We end this section with the Sparling-Nester-Witten identity. Given two spinor fields λ_A and $\mu_{A'}$, we can define a two-form by $L = -i\mu_{A'} \mathbf{d}\lambda_A \theta^{AA'}$. The \mathbf{d} here is the covariant exterior derivative of the four-dimensional spin connection acting on spinor valued differential forms, the $\theta^{AA'}$ are essentially the van-der-Waerden symbols and the product between differential forms is understood to be the Grassmann product. Computing the exterior derivative of L we find

$$dL = \mathbf{d}L = -i\mathbf{d}\mu_{A'} \mathbf{d}\lambda_A \theta^{AA'} - i\mu_{A'} \mathbf{d}^2 \lambda_A \theta^{AA'} = S + E. \quad (2.11)$$

The three-form E contains only curvature terms and, remarkably, only in the form of the Einstein tensor. In fact, one can show that (e.g., [2]) $E = -\frac{1}{2}V^a G_a{}^b \Sigma_b$ with $V^a = \lambda^A \mu^{A'}$ and $\Sigma_a = \frac{1}{6}e_{abcd}\theta^b\theta^c\theta^d$. Using the Einstein equations in the form $G_{ab} = -8\pi G T_{ab}$ the identity becomes

$$dL = -i\mathbf{d}\mu_{A'} \mathbf{d}\lambda_A \theta^{AA'} + 4\pi G V^a T_a{}^b \Sigma_b. \quad (2.12)$$

This identity is the basis for the spinorial proofs of positivity of mass in General Relativity which are obtained by choosing the spinor fields appropriately.

3 The integral formula

Consider the integral defined by

$$I(s) = \oint_{S_s} L. \quad (3.1)$$

We are interested in the change of I with the parameter s . To compute it we choose a vector field Z^a on Σ so that $Z^a \nabla_a s = 1$. Such vector fields exist because $\frac{\partial}{\partial s}$ is such a vector field but we could just as well choose any other field

which differs from it by terms tangent to the foliation. Now recall that for any two-form ω we have

$$\frac{d}{ds} \oint \omega = \oint L_Z \omega = \oint i_Z d\omega. \quad (3.2)$$

Here we have used the formula that the Lie derivative of a differential form is $L_Z \omega = di_Z \omega + i_Z d\omega$ and the fact that the two-surfaces are closed. This formula also shows that we can assume without loss of generality that the vector field Z^a is orthogonal to the foliation: components of Z^a tangent to the two-surfaces S_s would correspond to applying an infinitesimal diffeomorphism to the integrand which does not change the value of the integral because there are no boundaries. Now the identity (2.11) implies

$$\frac{d}{ds} \oint L = \oint i_Z dL = \oint i_Z S + \oint i_Z E. \quad (3.3)$$

The last term in this formula is easily evaluated. Note that $i_Z \theta^a = Z^a$ and $i_Z \Sigma_a = \frac{1}{2} e_{abcd} Z^b \theta^c \theta^d$ which, after restriction to the two-surface, becomes

$$3p_{[ab} m_{cd]} Z^b \theta^c \theta^d = p_{ab} Z^b d^2 A. \quad (3.4)$$

Thus,

$$\oint i_Z E = 4\pi G \oint (T_a{}^b V^a p_{bc} Z^c) d^2 A. \quad (3.5)$$

The evaluation of the first term in (3.3) is more complicated. We have

$$\begin{aligned} i_Z S &= i_Z \left(-i \mathbf{d} \mu_{A'} \mathbf{d} \lambda_A \theta^{AA'} \right) \\ &= -i Z^e \nabla_e \mu_{A'} \mathbf{d} \lambda_A \theta^{AA'} + i Z^e \nabla_e \lambda_A \mathbf{d} \mu_{A'} \theta^{AA'} - i \mathbf{d} \mu_{A'} \mathbf{d} \lambda_A Z^{AA'} \end{aligned} \quad (3.6)$$

After restriction to the surface the \mathbf{d} becomes the exterior covariant differential with respect to the two-dimensional Sen connection and we can express it in terms of the intrinsic spin connection using the relation $\mathbf{d} \lambda = \partial \lambda_A - Q_A{}^C \lambda_C$ where we denote the exterior covariant differential with respect to the intrinsic spin connection by ∂ and where $Q_A{}^B = \theta^a Q_{aA}{}^B$. Then the last term in (3.6) becomes

$$\begin{aligned} \mathbf{d} \mu_{A'} \mathbf{d} \lambda_A Z^{AA'} &= \left(\partial \mu_{A'} - \overline{Q}_{A'}{}^{C'} \mu_{C'} \right) \left(\partial \lambda_A - Q_A{}^C \lambda_C \right) Z^{AA'} \\ &= \partial \mu_{A'} \partial \lambda_A Z^{AA'} - \partial \mu_{A'} Q_A{}^C \lambda_C Z^{AA'} \\ &\quad + \partial \lambda_A \overline{Q}_{A'}{}^{C'} \mu_{C'} Z^{AA'} - Q_A{}^C \lambda_C \overline{Q}_{A'}{}^{C'} \mu_{C'} Z^{AA'}. \end{aligned} \quad (3.7)$$

The first term above can be rewritten as follows

$$\begin{aligned} \partial \mu_{A'} \partial \lambda_A Z^{AA'} &= \frac{1}{2} d \left(\mu_{A'} \partial \lambda_A Z^{AA'} - \lambda_A \partial \mu_{A'} Z^{AA'} \right) \\ &\quad + \frac{1}{2} \left\{ \lambda_A \partial^2 \mu_{A'} Z^{AA'} - \mu_{A'} \partial^2 \lambda_A Z^{AA'} - \lambda_A \partial \mu_{A'} \partial Z^{AA'} + \mu_{A'} \partial \lambda_A \partial Z^{AA'} \right\}. \end{aligned} \quad (3.8)$$

On the right hand side of this equation, the total derivative term can be dropped because it does not contribute to the integral. Hence we get after collecting similar terms

$$\begin{aligned}
i_Z S = & -i \left\{ -Z^e \nabla_e \mu_{A'} Q_A^E \lambda_E \theta^{AA'} + Z^e \nabla_e \lambda_A \bar{Q}_{A'}^{E'} \mu_{E'} \theta^{AA'} \right\} \\
& -i \left\{ \bar{Q}_{A'}^{E'} Q_A^E \mu_{E'} \lambda_E Z^{AA'} + \frac{1}{2} \lambda_A \partial^2 \mu_{A'} Z^{AA'} - \frac{1}{2} \mu_{A'} \partial^2 \lambda_A Z^{AA'} \right\} \\
& -i \left\{ \frac{1}{2} \mu_{A'} \partial \lambda_A \partial Z^{AA'} - \bar{Q}_{A'}^{E'} \mu_{E'} \partial \lambda_A Z^{AA'} + Z^e \nabla_e \mu_{A'} \partial \lambda_A \theta^{AA'} \right\} \\
& -i \left\{ -\frac{1}{2} \lambda_A \partial \mu_{A'} \partial Z^{AA'} + Q_A^E \lambda_E \partial \mu_{A'} Z^{AA'} - Z^e \nabla_e \lambda_A \partial \mu_{A'} \theta^{AA'} \right\}. \quad (3.9)
\end{aligned}$$

This is the general form of this term. Further simplifications can only be obtained by specialising the spinor fields $\mu_{A'}$ and λ_A . Usually this is done by imposing the Witten equation on the spinor fields. Here, we will proceed somewhat differently. The discussion is simplified by the following trivial observations:

- ∂ leaves the eigenspaces of γ_A^B invariant,
- ∂ leaves the splitting of the tangent space at points of S_s into vertical and horizontal parts invariant,
- Q_A^B interchanges the eigenspaces of γ_A^B ,
- $V^a = \lambda^A \mu^{A'}$ is horizontal/vertical iff λ^A and $\mu^{A'}$ are in different/same eigenspace(s).

Let us now assume that the spinor fields are both in the eigenspace of γ_A^B spanned by o_A , hence $\lambda_A = \lambda o_A$ and $\mu_{A'} = \bar{\mu} o_{A'}$. In this case, the flagpoles of both λ_A and $\mu_{A'}$ point along the generators of the same outgoing null hypersurface \mathcal{N}^+ . From the properties of the intrinsic derivative we find

$$\bar{Q}_{A'}^{E'} \mu_{E'} \partial \lambda_A Z^{AA'} = 0 = Q_A^E \lambda_E \partial \mu_{A'} Z^{AA'}. \quad (3.10)$$

We further assume that Z^a is orthogonal to the surfaces S_s so that we can write it as $Z^a = Z l^a + Z' n^a$. Using (2.10) and similar expansions for $\mu_{A'}$ and $Z^{AA'}$ we obtain

$$\begin{aligned}
i_Z S = & \lambda \rho \left(\iota^{A'} Z^e \nabla_e \mu_{A'} \right) d^2 A + \bar{\mu} \rho \left(\iota^A Z^e \nabla_e \lambda_A \right) d^2 A \\
& + \frac{1}{8} Z' \bar{\mu} \lambda \mathcal{R} d^2 A + Z \bar{\mu} \lambda (\sigma \bar{\sigma} - \rho^2) d^2 A \\
& + \frac{1}{2} (\bar{\mu} \bar{\partial}' \lambda \bar{\partial} Z' - \bar{\mu} \bar{\partial} \lambda \bar{\partial}' Z' - \lambda \bar{\partial}' \bar{\mu} \bar{\partial} Z' + \lambda \bar{\partial} \bar{\mu} \bar{\partial}' Z' - 2 \bar{\tau} \bar{\mu} \bar{\partial} \lambda Z' - 2 \tau \lambda \bar{\partial}' \bar{\mu} Z') d^2 A. \quad (3.11)
\end{aligned}$$

In this formula the last term can be simplified considerably by appropriate choice of the scalings of the spinor fields. First, we note that we can choose them completely freely on each surface S_s . Furthermore, we can exploit the fact that the vector field Z_a is hypersurface orthogonal. This leads to the equations

$$\bar{\partial} Z + \bar{\tau}' Z = 0, \quad \bar{\partial} Z' + \tau Z' = 0. \quad (3.12)$$

Inserting these into the last term in (3.11) yields the expression

$$-\frac{1}{2}Z'\bar{\mu}\lambda\left\{\tau\left(\frac{\partial'\lambda}{\lambda}+\frac{\partial'\bar{\mu}}{\bar{\mu}}\right)+\bar{\tau}\left(\frac{\partial\lambda}{\lambda}+\frac{\partial\bar{\mu}}{\bar{\mu}}\right)\right\}d^2A \quad (3.13)$$

for that term. As pointed out in the previous section, we can assume without loss of generality that the conormal of \mathcal{N}^- is a gradient. If λ_A is scaled so that its flag-pole coincides on Σ with that gradient then the following equations must hold there:

$$\eth(\lambda\bar{\lambda}) = \tau\lambda\bar{\lambda}, \quad (3.14)$$

$$\flat(\lambda\bar{\lambda}) = 0. \quad (3.15)$$

Obviously, these are conditions only for the extent $\lambda\bar{\lambda}$ of the flag-pole of the spinor field. The first equation above tells us how to fix it on S_s while the second equation specifies how to propagate it along the generators of \mathcal{N}^- . Note, that if we choose $\mu_{A'} = \bar{\lambda}_{A'}$ on Σ then (3.13) simplifies further by means of (3.14) to yield the final expression

$$-Z'\lambda\bar{\lambda}(\tau\bar{\tau})d^2A. \quad (3.16)$$

Henceforth, we will assume that the spinor fields have been chosen in that way on Σ . Therefore, (3.14) holds on Σ . We will not impose (3.15) because it is not needed. Note further, that there is some freedom left in choosing λ . First of all, it can be multiplied by any phase function on Σ and, what is more important, it is defined only up to the multiplication with any function $f(s)$ on Σ which is constant on the surfaces S_s .

Let us now consider the first two terms in (3.11). These can be expanded to yield the expression

$$\rho(Z^e\nabla_e(\lambda\bar{\lambda}) + Z\lambda\bar{\lambda}(\epsilon + \bar{\epsilon}) + Z'\lambda\bar{\lambda}(\gamma + \bar{\gamma}))d^2A, \quad (3.17)$$

where we have introduced the spin-coefficients ϵ and γ . However, from the definition of the spin-frame these coefficients are seen to vanish because on Σ we have $Do^A = 0$ and $D'\iota^A = 0$.

This concludes the evaluation of the right hand side of (3.3) in the case where both spinor fields are chosen to be tangent to the same null hypersurface. In this case, we obtain for the left hand side of (3.3)

$$-i\frac{d}{ds}\oint\mu_{A'}\mathbf{d}\lambda_A\theta^{AA'} = \frac{d}{ds}\oint\lambda\bar{\lambda}\rho d^2A. \quad (3.18)$$

So we can finally put everything together to obtain the integral formula

$$\begin{aligned} \frac{d}{ds}\oint\phi\rho d^2A &= \oint\rho\dot{\phi}d^2A + Z'\phi\oint\left(\frac{\mathcal{R}}{8} - \tau\bar{\tau}\right)d^2A \\ &+ \oint Z\phi(\sigma\bar{\sigma} - \rho^2)d^2A + 4\pi G\oint\phi(l^aT_a{}^bp_{bc}Z^c)d^2A, \end{aligned} \quad (3.19)$$

where ϕ is a positive $(-1, -1)$ -function on Σ which satisfies the equation $\eth\phi = \tau\phi$ on each two-surface S_s and $\dot{\phi} = Z^e\nabla_e\phi$. Note, that this equation always has solutions because it corresponds to the geometric statement that one can always

choose the tangents of the outgoing null geodesics on S_s to coincide with the gradients of the null hypersurface which is generated by them. These tangent vectors are given by ϕl^a . We have pulled out the factor $Z'\phi$ from the integral because, as a consequence of (3.12) and (3.14), we have $Z'\phi = \text{const.}$ on the surfaces S_s . This equality corresponds to the fact that the function which defines the null hypersurface foliation can be chosen to agree with s on Σ (cf. the discussion at the beginning of section 2). Note, that we have not yet specified the scaling of the spin-frame. Under special circumstances this can be chosen to further simplify the formula. Furthermore, the term involving the scalar curvature integrates to a constant by the Gauß-Bonnet theorem.

The quantity on the left hand side is the integrated divergence of that null geodesic congruence emanating from S_s for which the tangent vectors coincide with the gradient of the null hypersurface that is formed by the congruence. On the right hand side we find terms which have to do with the geometry of the two-surfaces, namely the scalar curvature \mathcal{R} and the determinant $\rho^2 - \sigma\bar{\sigma}$ of the extrinsic curvature with respect to the corresponding null normal. Note, that this expression occurs in the Gauß maps defined in [4] and has to do with the convexity of the two-surfaces. The τ -terms and the term involving the derivative of ϕ have no immediate geometric meaning. However, it is worth to point out that on a spacelike hypersurface Σ under the assumption of the dominant energy condition the τ -term combines with the energy-momentum term with the same sign. So it might be possible to view that term as some kind of gravitational contribution to the total energy. The term involving the derivative of ϕ is present because it guarantees the invariance under reparametrisation of the foliation.

In the case where the spinor fields are aligned along the other null hypersurface \mathcal{N}^+ (so that $\lambda_A = \bar{\mu}_A = \bar{\mu}\iota_A$) we use the analogous assumptions to arrive at the formula

$$\begin{aligned} \frac{d}{ds} \oint \phi' \rho' d^2 A &= \oint \rho' \dot{\phi}' d^2 A + Z \phi' \oint \left(\frac{\mathcal{R}}{8} - \tau' \bar{\tau}' \right) d^2 A \\ &+ \oint Z' \phi' \left(\sigma' \bar{\sigma}' - \rho'^2 \right) d^2 A - 4\pi G \oint \phi' \left(n^a T_a{}^b p_{bc} Z^c \right) d^2 A, \end{aligned} \quad (3.20)$$

where now ϕ' is a $(1,1)$ -function satisfying $\bar{\partial}' \phi' = \tau' \phi'$ on each two-surface S_s .

4 Special cases

4.1 On a null hypersurface

First, we look at the case where Σ is itself a null hypersurface. Let l_a be the normal to Σ scaled so that it is a gradient. Then l^a is tangent to the null geodesics generating Σ and we choose an affine parameter s along each generator. Fix a two-dimensional surface S orthogonal to each generator and define a foliation of Σ by the surfaces S_s of constant affine parameter s with $S = S_0$. Thus we can take $\phi = 1$ and $Z^a = l^a$, i.e., $Z' = 0$ and $Z = 1$. Then (3.19) results in the formula

$$\frac{d}{ds} \oint \rho d^2 A = \oint (\sigma \bar{\sigma} - \rho^2) d^2 A + 4\pi G \oint T_{ab} l^a l^b d^2 A. \quad (4.1)$$

This formula is one of the optical equations [5] integrated over S_s . It is the equation which governs the focusing of a bundle of light rays. This can be seen by noting that the left hand side can be rewritten as

$$\frac{d}{ds} \oint \rho d^2 A = \oint (D\rho - 2\rho^2) d^2 A. \quad (4.2)$$

by an argument similar to equation (3.2). Thus we obtain

$$\oint D\rho d^2 A = \oint (\rho^2 + \sigma\bar{\sigma}) d^2 A + 4\pi G \oint T_{ab} l^a l^b d^2 A. \quad (4.3)$$

The formula (3.20) can be specialised in this case to yield

$$\frac{d}{ds} \oint \rho' d^2 A = \oint \left(\frac{1}{8} \mathcal{R} - \tau' \bar{\tau}' \right) d^2 A - 4\pi G \oint T_{ab} n^a l^b d^2 A. \quad (4.4)$$

Here, we have used the fact that ϕ' is constant on the two-surfaces because $Z = 1$.

4.2 Spherical symmetry

The next special case occurs in space-times with spherical symmetry. We take Σ as a spacelike hypersurface which is foliated by the spheres S_s of symmetry. Due to the symmetry all quantities with a non-vanishing spin-weight are zero and, in addition, all integrands are constant on S_s so that the integrals yield the integrand times the area $A(s) = 4\pi R(s)^2$ of the surface. Then (3.19) gives

$$\frac{d}{ds} (\phi \rho R^2) = \left(\rho \dot{\phi} + \frac{1}{8} \mathcal{R} \phi Z' \right) R^2 - Z \phi (\rho R)^2 + 4\pi G \phi (l^a T_a{}^b p_{bc} Z^c) R^2. \quad (4.5)$$

Since we are free to multiply ϕ by any function of s , we may take $\phi = 1/R$. Next, we take the parameter s to be the proper length along a radial geodesic so that Z^a is a unit vector field. Furthermore, we choose the spin-frame so that $Z' = -Z = 1/\sqrt{2}$, thus breaking the scaling invariance (2.6). Then we also have $p_{bc} Z^c = Z l_b - Z' n_b = -Z'(l_b + n_b) = -t^b$ where t_b is the future pointing timelike unit normal vector to Σ . Finally, inserting the scalar curvature $\mathcal{R} = 4/R^2$ of the surface we obtain

$$\frac{d}{ds} (\rho R) = \frac{1}{\sqrt{2}R} \left(\frac{1}{2} + (\rho R)(\rho' R) \right) - 4\pi G R T_{ab} l^a t^b. \quad (4.6)$$

Here, we have used the formula $Z^e \nabla_e R = (\rho - \rho')R/\sqrt{2}$, a consequence of the formula for the rate of change in the area element along a vector field [2]. Let us now define the quantities $\omega_+ = 2\sqrt{2}\rho R$ and $\omega_- = -2\sqrt{2}\rho' R$ then we can establish complete agreement with [6] where an alternative form of the constraint equations for spherically symmetric space-times has been derived. These equations are

$$\frac{d\omega_+}{ds} = \frac{1}{4R} (4 - \omega_+ \omega_-) - 8\pi G R T_{ab} (\sqrt{2} l^a) t^b, \quad (4.7)$$

$$\frac{d\omega_-}{ds} = \frac{1}{4R} (4 - \omega_+ \omega_-) - 8\pi G R T_{ab} (\sqrt{2} n^a) t^b. \quad (4.8)$$

These two equations have several important consequences. First, assuming that Σ is topologically \mathbb{R}^3 , asymptotically flat and regularity at $R = 0$ of Σ and the dominant energy condition one can deduce global bounds for the quantities ω_{\pm} . These in turn imply statements about the occurrence of trapped surfaces and singularities in the domain of dependence of Σ .

Furthermore, one can easily prove the Penrose inequality in this case [7]. Suppose that Σ contains an apparent horizon and is asymptotically flat. Consider the Hawking mass for a sphere S_s which is given in the present context by

$$m_H(s) = \frac{R}{8G} (4 - \omega_+ \omega_-). \quad (4.9)$$

Its derivative along Z^a is

$$\frac{dm_H}{ds} = \pi R^2 T_{ab} l^b (l^a \omega_- + n^a \omega_+). \quad (4.10)$$

At the outermost apparent horizon with area $A_0 = 4\pi R_0^2$ the mass is $m_H = R_0/(2G)$ while at spatial infinity we have $m_H = m_{ADM}$. In the region in between both ω_{\pm} are positive. From (4.10) we find that

$$\frac{dm_H}{dR} = 4\pi R^2 T_{ab} l^b H^a \geq 0. \quad (4.11)$$

The positivity follows from the fact that under the conditions stated above

$$H^a = \frac{\omega_-}{\omega_+ + \omega_-} l^a + \frac{\omega_+}{\omega_+ + \omega_-} n^a \quad (4.12)$$

is a future pointing timelike vector so that the sign is a consequence of the dominant energy condition. From this we obtain the inequality $R_0 \leq 2Gm_{ADM}$ or

$$A_0 \leq 16\pi G^2 m_{ADM}^2. \quad (4.13)$$

4.3 At null infinity

Our last specialisation occurs in asymptotically flat space-times which admit a regular null infinity \mathcal{J} when Σ coincides with \mathcal{J} . To treat this case we proceed as follows. We introduce Bondi coordinates $(u, r, \zeta, \bar{\zeta})$ and the usual Bondi null tetrad in a neighbourhood of \mathcal{J} . We evaluate the integral formula on the hypersurfaces Σ_r of constant r and then take the limit $r \rightarrow \infty$. The evaluation of the formulae has to be done only to leading order in $1/r$. The asymptotic solution of the vacuum equations is taken from [8].

Each hypersurface Σ_r is foliated by its intersections S_u with the null hypersurfaces $u = \text{const}$ so we take $s = u$. Since the Bondi tetrad is not yet adapted to these surfaces we need to perform a null rotation around l^a to put n^a orthogonal to S_u . Then the covariant form of the tangent vectors to the outgoing null hypersurfaces is

$$l = du, \quad n = dr - (U - \omega\bar{\omega})du. \quad (4.14)$$

The spin coefficients need to be changed accordingly. In particular, we get to leading order

$$\rho = -r^{-1} + O(r^{-3}), \quad (4.15)$$

$$\rho' = \frac{1}{2}r^{-1} + \left(\Psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0 + \bar{\sigma}^2 \bar{\sigma}^0 \right) r^{-2} + O(r^{-3}), \quad (4.16)$$

$$\sigma = \sigma^0 r^{-2} + O(r^{-3}), \quad (4.17)$$

$$\sigma' = -\dot{\bar{\sigma}}^0 r^{-1} + O(r^{-2}), \quad (4.18)$$

$$\tau' = -\bar{\sigma}' \sigma^0 r^{-2} + O(r^{-3}). \quad (4.19)$$

Here, $\bar{\sigma}$ is defined with respect to the unit sphere. Note, that ρ' is real due to the null rotation and by virtue of an equation on \mathcal{J} which follows from the asymptotic solution of the vacuum field equations.

The vector field Z^a is given by $Z^a = n^a - (U - \omega \bar{\omega}) l^a$ and the function ϕ' on Σ_r can be taken to have the form $\phi' = \phi_0(u) + \phi_1(u)r^{-1} + O(r^{-2})$, where $\bar{\sigma}' \phi_1 = (\bar{\sigma}' \sigma^0) \phi_0$. Finally, the scalar curvature of the two-surfaces S_u is $\mathcal{R} = 4r^{-2} + O(r^{-3})$. Inserting these expansions into (3.20) gives

$$\begin{aligned} \frac{d}{du} \oint \left\{ \frac{1}{2} (r\phi_0 + \phi_1) + \left(\Psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0 + \bar{\sigma}^2 \bar{\sigma}^0 \right) \phi_0 \right\} d^2\Omega = \\ \oint \frac{1}{2} (r\dot{\phi}_0 + \dot{\phi}_1) + \left(\Psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0 + \bar{\sigma}^2 \bar{\sigma}^0 \right) \dot{\phi}_0 d^2\Omega \\ + \oint \frac{1}{4} \phi_0 d^2\Omega + \oint \phi_0 \left(\dot{\bar{\sigma}}^0 \dot{\sigma}^0 - \frac{1}{4} \right) d^2\Omega + O(r^{-1}). \end{aligned} \quad (4.20)$$

Here $d^2\Omega$ is the surface element of the unit sphere. This equation can be simplified. Note, that the $O(r)$ -terms cancel and that the $\bar{\sigma}^2 \bar{\sigma}^0$ term vanishes upon integration over S_u so that in the limit $r \rightarrow \infty$ we obtain the Bondi mass loss formula

$$\frac{d}{du} m_B = \frac{d}{du} \left\{ -\frac{1}{4\pi G} \oint \Psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0 d^2\Omega \right\} = -\frac{1}{4\pi G} \oint \dot{\bar{\sigma}}^0 \dot{\sigma}^0 d^2\Omega. \quad (4.21)$$

The other integral formula (3.20) contains no information when Σ coincides with \mathcal{J} .

It is worthwhile to point out that in the limit when Σ coincides with \mathcal{J} it is a null hypersurface and that the integral formula is then exactly the one discussed as the first special case, namely the integrated focusing equation. This seems to suggest that the gravitational energy flux and focusing power are really closely related. This has been discussed before [9] and there have also been attempts to prove positivity of mass from the focusing properties of light rays [10].

5 Discussion

We have derived an integral formula valid on an arbitrary hypersurface Σ in a space-time which satisfies Einstein's equation. It is obtained in a straightforward way from the Sparling-Nester-Witten identity by a particular choice of the spinor fields. In contrast to the usual procedure of fixing the spinor fields by having them obey a differential equation here they are chosen in a geometric way. It is

not clear whether this fixing is optimal. It might well be that there is a different choice which eliminates the somewhat arbitrary and uncontrollable functions ϕ and ϕ' . Furthermore, in the final form there is no essentially spinorial feature left in the integral formula. It might be worthwhile to see whether one can derive the formula without using spinors.

Of course, there are many questions left. We have seen that the integral formula reduces to the special form of the constraint equations derived by [6] in the case of spherical symmetry with spacelike Σ . They can derive bounds for the optical scalars ω_{\pm} when Σ is a regular embedded hypersurface which allows them to obtain statements about the occurrence of trapped surfaces and singularities in the domain of dependence of Σ . Is it possible to get bounds on the integral expressions or some variant of those, too? Can one prove the Penrose inequality using these integral formulae in a way similar to the one given above?

We have not looked at the other possibilities to choose the spinor fields. The cases where the two spinor fields are aligned along different null hypersurfaces do not seem to lead to any similarly nice integral formula.

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